

$$\begin{aligned}
& \lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \lim_{\Delta z \rightarrow 0} \sum_i \sum_j \sum_k \Delta x \Delta y \Delta z \rho(x_i, y_j, z_k) \\
= & \lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \sum_i \sum_j \Delta x \Delta y \left[ \lim_{\Delta z \rightarrow 0} \sum_k \Delta z \rho(x_i, y_j, z_k) \right] \\
= & \lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \sum_i \sum_j \Delta x \Delta y \int dz \rho(x_i, y_j, z) \\
= & \lim_{\Delta x \rightarrow 0} \sum_i \Delta x \left[ \lim_{\Delta y \rightarrow 0} \sum_j \Delta y \int dz \rho(x_i, y_j, z) \right] \\
= & \lim_{\Delta x \rightarrow 0} \sum_i \Delta x \left[ \int dy \int dz \rho(x_i, y, z) \right] \\
= & \int dx \int dy \int dz \rho(x, y, z)
\end{aligned}$$